# *THE LADDER HYPOTHESIS*

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## *CREDITS*

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## *ABSTRACT*

In a paper from 18.aug 2001 available at www.islandia.is/gko/010818.pdf, a thermoelectric generator was constructed from a large number of series connected parallelepipeds. The hot and/or cold reservoir was made of some electrically conductive metal, and the fluid was to some extent conductive to the electrical ground. This topology generated a number of small capacitors, each formed by two parallelepiped crystal faces and the grounded thermal reservoir. When analysing the frequency behaviour of such a device, rational polynomials manifested themselves and proved to be a rich source of advanced mathematical relations.

## *INTRODUCTION*

Electronic network theory of linear circuit elements has a strong connection to the mathematics and the algebra of Polynomials in the Complex Domain. This is due to the fact that a dynamical differential equation can be Laplace transformed from the time domain into the frequency domain, and in the process, is turned into a polynomial in the complex variable "$s = \beta + i\omega$", where ($\omega$) is the angular frequency "$f = \omega/2\pi$" and ($\beta$) is the dissipative (or generative) time-constant. A thermoelectric generator consisting of a large number of small crystals connected serially together can be considered as a naturally occurring network of lumped elements, and is ideally suited to network analyses in the frequency domain.

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## 1. The "Ladder Hypothesis" summarized:

The coefficients of the numerator and the denominator polynomials, expressing the electric impedance of series connected pi-networks as seen in the paper www.islandia.is/gko/010818.pdf, can be arranged in two Pascal's-like triangles:

$$\text{Denominator:}\qquad \begin{matrix} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 1 & & 3 & & 1 & & & \\ & & 1 & & 6 & & 5 & & 1 & & \\ & 1 & & 10 & & 15 & & 7 & & 1 & \\ 1 & & 15 & & 35 & & 28 & & 9 & & 1 \end{matrix}$$

$$\text{Numerator:}\qquad \begin{matrix} & & & & & 1 & & & & & \\ & & & & 2 & & 1 & & & & \\ & & & 3 & & 4 & & 1 & & & \\ & & 4 & & 10 & & 6 & & 1 & & \\ & 5 & & 20 & & 21 & & 8 & & 1 & \\ 6 & & 35 & & 56 & & 36 & & 10 & & 1 \end{matrix}$$

Let D(n,m) and N(n,m) represent the denominator and numerator coefficients respectively, where (n) is the number of pi-networks and (m) is the coefficient's index from left to right.

$$D(n,1)=1$$
$$D(n,2)=\tfrac{1}{2}\cdot n\cdot(n-1)$$
$$D(n,3)=\tfrac{1}{24}\cdot n\cdot(n^2-1)\cdot(n-2)$$
$$D(n,m)=\sum_{k=1}^{n-1}N(k,m-1)$$
$$D(n,n-1)=2\cdot n-3$$
$$D(n,n)=1$$
$$D(n,m+1)=\frac{n^2-m^2-n+m}{4\cdot m\cdot(m-\frac{1}{2})}\cdot D(n,m)$$

$$N(n,1)=n$$
$$N(n,2)=\tfrac{1}{6}\cdot n\cdot(n^2-1)$$
$$N(n,3)=\tfrac{1}{120}\cdot n\cdot(n^2-1)\cdot(n^2-2^2)$$
$$N(n,m)=\sum_{k=1}^{n}D(k,m)$$
$$N(n,n-1)=2\cdot n-2$$
$$N(n,n)=1$$
$$N(n,m+1)=\frac{n^2-m^2}{4\cdot m\cdot(m+\frac{1}{2})}\cdot N(n,m)$$

The two solutions to the recursive equations above were found to be the following two repeated products:

$$D(n,m)=\frac{n\cdot(n+1-m)}{(2\cdot m-2)!}\cdot\prod_{k=1}^{m-2}(n^2-k^2)=\frac{n+1-m}{n\cdot(2\cdot m-2)!}\cdot\prod_{k=0}^{m-2}(n^2-k^2)$$

$$N(n,m)=\frac{n}{(2\cdot m-1)!}\cdot\prod_{k=1}^{m-1}(n^2-k^2)=\frac{1}{n\cdot(2\cdot m-1)!}\cdot\prod_{k=0}^{m-1}(n^2-k^2)$$

As stated before, these are the coefficients in the numerator- and the denominator polynomials for the LADDER:

$$Q_n(a)=1+\tfrac{1}{2}\cdot a\cdot n\cdot(n-1)+\tfrac{1}{24}\cdot a^2\cdot n\cdot(n^2-1)\cdot(n-2)+\cdots+a^{m-1}\cdot D(n,m)+\cdots+a^{n-1}$$

$$P_n(a)=n+\tfrac{1}{6}\cdot a\cdot n\cdot(n^2-1)+\tfrac{1}{120}\cdot a^2\cdot n\cdot(n^2-1)\cdot(n^2-2^2)+\cdots+a^{m-1}\cdot N(n,m)+\cdots+a^{n-1}$$

The Ladder Polynomials are recursively related and their properties can be summarised by the following lists:

$$Q_1(a)=1$$
$$Q_n(a)=Q_{n-1}(a)+a\cdot P_{n-1}(a)$$
$$Q_n(a)=1+a\cdot\sum_{k=1}^{n-1}P_k(a)$$
$$Q_n(a)=\prod_{k=1}^{n-1}\left(a+4\cdot\sin^2\left(\frac{\pi\cdot(2k-1)}{2\cdot(2n-1)}\right)\right)$$

$$P_1(a)=1$$
$$P_n(a)=Q_{n-1}(a)+(1+a)\cdot P_{n-1}(a)$$
$$P_n(a)=1+P_{n-1}(a)+a\cdot\sum_{k=1}^{n-1}P_k(a)$$
$$P_n(a)=\prod_{k=1}^{n-1}\left(a+4\cdot\sin^2\left(\frac{\pi\cdot k}{2\cdot n}\right)\right)$$

## 2. Finite products of the Sinus function:

By inserting "a=0" into the factored polynomials $Q_n$ and $P_n$, two statements are derivable from "$Q_n(0) = 1$" and "$P_n(0) = n$":

$$\sin\left(\tfrac{\pi}{2n}\right)\cdot\sin\left(\tfrac{2\pi}{2n}\right)\cdot\sin\left(\tfrac{3\pi}{2n}\right)\cdots\sin\left(\tfrac{\pi\cdot(n-1)}{2n}\right)=\prod_{k=1}^{n-1}\sin\left(\frac{k\cdot\pi}{2\cdot n}\right)=2^{1-n}\cdot\sqrt{n}$$

$$\sin\left(\tfrac{\pi}{4n-2}\right)\cdot\sin\left(\tfrac{3\pi}{4n-2}\right)\cdot\sin\left(\tfrac{5\pi}{4n-2}\right)\cdots\sin\left(\tfrac{\pi\cdot(2n-3)}{4n-2}\right)=\prod_{k=1}^{n-1}\sin\left(\frac{\pi\cdot(2k-1)}{2\cdot(2n-1)}\right)=2^{1-n}$$

A simpler version is known in the literature*, the Fundamental Sinus Product. It has recently gained some attention*:

$$\sin\left(\tfrac{\pi}{n}\right)\cdot\sin\left(\tfrac{2\pi}{n}\right)\cdot\sin\left(\tfrac{3\pi}{n}\right)\cdots\sin\left(\tfrac{\pi\cdot(n-1)}{n}\right)=\prod_{k=1}^{n-1}\sin\left(\frac{\pi\cdot k}{n}\right)=2^{1-n}\cdot n$$

Even integers "n = 2n" and the 90° symmetry of sinus will transform this into the half angle sinus product squared:

$$\sin\left(\tfrac{\pi}{2n}\right)\cdot\sin\left(\tfrac{2\pi}{2n}\right)\cdot\sin\left(\tfrac{3\pi}{2n}\right)\cdots\sin\left(\tfrac{\pi\cdot(n-1)}{2n}\right)\cdot 1\cdot\sin\left(\tfrac{\pi\cdot(n-1)}{2n}\right)\cdot\sin\left(\tfrac{\pi\cdot(n-2)}{2n}\right)\cdots\sin\left(\tfrac{\pi}{2n}\right)=\left(\prod_{k=1}^{n-1}\sin\left(\frac{\pi\cdot k}{2n}\right)\right)^2=4^{1-n}\cdot n$$

By the same method, we can generate a relation for the odd integers (2n-1) as seen in the next section (3).

## 3. The finite sums of Logarithms of Sinus functions:

Now let n>1 and take the natural logarithm of all four products of sinus to get four equally interesting statements:

$$\prod_{k=1}^{n-1}2\cdot\sin\left(\frac{\pi\cdot k}{n}\right)=n \quad\Leftrightarrow\quad \sum_{k=1}^{n-1}\ln\sin\left(\frac{\pi\cdot k}{n}\right)=-(n-1)\cdot\ln(2)+\ln(n)$$

$$\prod_{k=1}^{n-1}2\cdot\sin\left(\frac{\pi\cdot k}{2n}\right)=\sqrt{n} \quad\Leftrightarrow\quad \sum_{k=1}^{n-1}\ln\sin\left(\frac{\pi\cdot k}{2n}\right)=-(n-1)\cdot\ln(2)+\frac{1}{2}\cdot\ln(n)$$

$$\prod_{k=1}^{n-1}2\cdot\sin\left(\frac{\pi\cdot k}{2n-1}\right)=\sqrt{2n-1} \quad\Leftrightarrow\quad \sum_{k=1}^{n-1}\ln\sin\left(\frac{\pi\cdot k}{2n-1}\right)=-(n-1)\cdot\ln(2)+\frac{1}{2}\cdot\ln(2n-1)$$

$$\prod_{k=1}^{n-1}2\cdot\sin\left(\frac{\pi\cdot(2k-1)}{4n-2}\right)=1 \quad\Leftrightarrow\quad \sum_{k=1}^{n-1}\ln\sin\left(\frac{\pi\cdot(2k-1)}{4n-2}\right)=-(n-1)\cdot\ln(2)$$

We acknowledge the fact, that the last statement has not been proved yet. It is a strong statement, being constant for any n.

## 4. Positive integers expressed as products of "zeros": $n=(1-1^{-1/n})(1-1^{-2/n})\ldots$

Some peculiar products involving only the number "1" and powers of it, can generate the integers, the square root of integers and much more... The Finite Products of sinuses simplify by the exponential substitution "$\sin x = (2i)^{-1}(e^{ix} - e^{-ix})$", also known as the hyperbolic sinus of an imaginary argument. As the arguments are harmonic in our case, we can easily derive:

$$\prod_{k=1}^{n-1}\left(1-1^{-k/n}\right)=n \qquad \prod_{k=1}^{n-1}\left(1-\mathrm{e}^{-\frac{i2\pi\cdot k}{n}}\right)=i^{n-1}\cdot\mathrm{e}^{-\frac{i\pi\cdot(n-1)}{2}}\cdot n$$

$$\prod_{k=1}^{n-1}\left(1-(-1)^{-k/n}\right)=\sqrt{n\cdot i^{(n-1)}} \qquad \prod_{k=1}^{n-1}\left(1-\mathrm{e}^{-\frac{i\pi\cdot k}{n}}\right)=i^{n-1}\cdot\mathrm{e}^{-\frac{i\pi\cdot(n-1)}{4}}\cdot\sqrt{n}=i^{(n-1)/2}\cdot\sqrt{n}$$

$$\prod_{k=1}^{n-1}\left(1-1^{-k/(2n-1)}\right)=i^{(n-1)^2/(2n-1)}\cdot\sqrt{2n-1} \qquad \prod_{k=1}^{n-1}\left(1-\mathrm{e}^{-\frac{i2\pi\cdot k}{2n-1}}\right)=i^{n-1}\cdot\mathrm{e}^{-\frac{i\pi\cdot n\cdot(n-1)}{2\cdot(2n-1)}}\cdot\sqrt{2n-1}$$

$$\prod_{k=1}^{n-1}\left(1-(-1)^{-(2k-1)/(2n-1)}\right)=i^{n\cdot(n-1)/(2n-1)} \qquad \prod_{k=1}^{n-1}\left(1-\mathrm{e}^{-\frac{i\pi\cdot(2k-1)}{2n-1}}\right)=i^{n-1}\cdot\mathrm{e}^{-\frac{i\pi\cdot(n-1)^2}{4n-2}}=\mathrm{e}^{\frac{i\pi\cdot n\cdot(n-1)}{4n-2}}$$

On the left we have a minimally expressed statement, on the right we have the essence of it's proof.

## 5. The finite product of Cosinus functions:

We would now like to make contact with the finite product of the Cosinus Function. This can be accomplished by using the identity sin 2x = 2 sinx cosx.

$$\prod_{k=1}^{n-1} 2\cdot\sin\left(\frac{\pi\cdot k}{n}\right)=\prod_{k=1}^{n-1} 4\cdot\sin\left(\frac{\pi\cdot k}{2n}\right)\cdot\cos\left(\frac{\pi\cdot k}{2n}\right)=\left(\prod_{k=1}^{n-1} 2\cdot\sin\left(\frac{\pi\cdot k}{2n}\right)\right)\cdot\left(\prod_{k=1}^{n-1} 2\cdot\cos\left(\frac{\pi\cdot k}{2n}\right)\right)=n$$

This rather surprising result can also be deduced from the identity sin (x) = cos (π/2 – x) and the Π sin(πk/2n) product:

$$\prod_{k=1}^{n-1} 2\cdot\sin\left(\frac{\pi\cdot k}{2n}\right)=\prod_{k=1}^{n-1} 2\cdot\cos\left(\frac{\pi\cdot(n-k)}{2n}\right)=\prod_{\ell=n-1}^{1} 2\cdot\cos\left(\frac{\pi\cdot\ell}{2n}\right)=\prod_{k=1}^{n-1} 2\cdot\cos\left(\frac{\pi\cdot k}{2n}\right)=\sqrt{n}$$

We can now add both cos(x) and tan(x) to our arsenal of products of trigonometric functions:

$$\prod_{k=1}^{n-1} 2\cdot\sin\left(\frac{\pi\cdot k}{2n}\right)=\prod_{k=1}^{n-1} 2\cdot\cos\left(\frac{\pi\cdot k}{2n}\right)=\sqrt{n}\ ,\qquad\Rightarrow\qquad\prod_{k=1}^{n-1}\tan\left(\frac{\pi\cdot k}{2n}\right)=1$$

From properties of the cos(x) function, we find that Π cos(πk/n) = {-1, 0, +1} depending on evenness and oddity of n. The zero comes from even n = 2,4,6,… the minus one from n = 3,7,11,… and plus one from n = 5,9,13,…

## 6. The Cotangent function and an infinite product expansion for Sinus:

The cot(x) function is rather special being the derivative of lnsin(x). The Riemann Zeta function ζ(s) appears here:

$$\cot x=\sum_{-\infty}^{+\infty}\frac{1}{\pi\cdot\ell+x}=x\cdot\sum_{-\infty}^{+\infty}\frac{1}{x^2-\pi^2\cdot\ell^2}=\frac{1}{x}-2x\cdot\sum_{\ell=1}^{+\infty}\frac{1}{\pi^2\cdot\ell^2-x^2}=\frac{1}{x}-\frac{2}{x}\cdot\sum_{k=1}^{\infty}\zeta(2k)\cdot\left(\frac{x}{\pi}\right)^{2k}$$

$$\ln\sin x=\ln x-\sum_{k=1}^{\infty}\frac{2^{2k}\cdot B_k\cdot x^{2k}}{2\cdot k\cdot(2k)!}=\ln x-\sum_{k=1}^{\infty}\frac{\zeta(2k)}{k}\cdot\left(\frac{x}{\pi}\right)^{2k}=\ln x-\sum_{k,\ell=1}^{\infty}\frac{1}{k}\cdot\left(\frac{x}{\pi\cdot\ell}\right)^{2k}$$

We have used the infinite sum definition for the Riemann Zeta function to arrive at the final infinite double-sum. This result can also be obtained directly from the infinite product formula for the sinus function:

$$\sin x=x\cdot\left(1-\frac{x^2}{\pi^2}\right)\cdot\left(1-\frac{x^2}{4\pi^2}\right)\cdot\left(1-\frac{x^2}{9\pi^2}\right)\cdots=x\cdot\prod_{\ell=1}^{\infty}\left(1-\frac{x^2}{\ell^2\cdot\pi^2}\right)$$

Notice the odd x in the sin function, which shows sin(x)/x as a simpler object than sin(x). Now take the natural logarithm of the infinite product for the sin function and get:

$$\ln\sin x=\ln x+\sum_{\ell=1}^{\infty}\ln\left(1-\left(\frac{x}{\pi\cdot\ell}\right)^2\right)=\ln x-\sum_{\ell,k=1}^{\infty}\frac{1}{k}\cdot\left(\frac{x}{\pi\cdot\ell}\right)^{2k}$$

The interval of convergence is unconditional, at least on the interval [-π < x < π], which is in fact the largest interval to occur. For clarity let us now summarise this result in a formal way:

$$\sum_{k=1}^{\infty}\frac{\zeta(2k)}{k}\cdot\left(\frac{x}{\pi}\right)^{2k}=\ln\left(\frac{x}{\sin x}\right)=-\ln\left(\frac{\sin x}{x}\right)\qquad,\ \zeta(2k)=\sum_{\ell=1}^{\infty}\ell^{-2k}=\frac{(2\pi)^{2k}}{(2k)!}\cdot\frac{B_k}{2}$$

The last equality support the recent attempts to redefine the Bernoulli numbers to be even indexed, as we have related them to very fundamental functions, the natural logarithm and the sinus. The first seven Bernoulli numbers are:

$$B_1=1/6,\quad B_2=1/30,\quad B_3=1/42,\quad B_4=1/30,\quad B_5=5/66,\quad B_6=691/2730,\quad B_7=7/6$$

It is interesting that all denominators above are divisible by 6. The zeroth Bernoulli number is calculable as $B_0$ = -1.

## 7. Some power series with Riemann Zeta coefficients:

To get a broader view on power series with Bernoulli numbers and/or the Riemann Zeta functions, let us reproduce some known results from the power series for tan(x) and cot(x):

$$\tan x = x + \frac{x^3}{3} + \frac{2\cdot x^5}{15} + \cdots + \frac{2^{2k}\cdot\left(2^{2k}-1\right)\cdot B_k \cdot x^{2k-1}}{(2k)!} + \cdots = \frac{2}{x}\cdot\sum_{k=1}^{\infty}\left(2^{2k}-1\right)\cdot\zeta(2k)\cdot\left(\frac{x}{\pi}\right)^{2k}$$

$$\cot x = \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \cdots - \frac{2^{2k}\cdot B_k\cdot x^{2k-1}}{(2k)!} + \cdots = \frac{1}{x}\cdot\left(1 - 2\cdot\sum_{k=1}^{\infty}\zeta(2k)\cdot\left(\frac{x}{\pi}\right)^{2k}\right)$$

At this moment, let us pause to express the first few even Riemann Zeta values and the alternating sign series also:

$$\zeta(2) = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$$

$$\zeta(4) = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots = \frac{\pi^4}{90}$$

$$\zeta(6) = \frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \cdots = \frac{\pi^6}{945}$$

$$\zeta(8) = \frac{1}{1^8} + \frac{1}{2^8} + \frac{1}{3^8} + \cdots = \frac{\pi^8}{9450}$$

$$\zeta(10) = \frac{1}{1^{10}} + \frac{1}{2^{10}} + \frac{1}{3^8} + \cdots = \frac{\pi^{10}}{93555}$$

$$\zeta(12) = \frac{1}{1^{12}} + \frac{1}{2^{12}} + \frac{1}{3^{12}} + \cdots = \frac{691\cdot\pi^{12}}{638512875}$$

$$\zeta(14) = \frac{1}{1^{14}} + \frac{1}{2^{14}} + \frac{1}{3^{14}} + \cdots = \frac{2\cdot\pi^{14}}{18243225}$$

$$\zeta^{\pm}(2) = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots = \frac{\pi^2}{12}$$

$$\zeta^{\pm}(4) = \frac{1}{1^4} - \frac{1}{2^4} + \frac{1}{3^4} - \cdots = \frac{7\cdot\pi^4}{720}$$

$$\zeta^{\pm}(6) = \frac{1}{1^6} - \frac{1}{2^6} + \frac{1}{3^6} - \cdots = \frac{31\cdot\pi^6}{30240}$$

$$\zeta^{\pm}(8) = \frac{1}{1^8} - \frac{1}{2^8} + \frac{1}{3^8} - \cdots = \frac{127\cdot\pi^8}{1\,209\,600}$$

$$\zeta^{\pm}(10) = \frac{1}{1^{10}} - \frac{1}{2^{10}} + \frac{1}{3^{10}} - \cdots = \frac{511\cdot\pi^{10}}{47\,900\,160}$$

$$\zeta^{\pm}(12) = \frac{1}{1^{12}} - \frac{1}{2^{12}} + \frac{1}{3^{12}} - \cdots = \frac{1\,414\,477\cdot\pi^{12}}{1\,307\,674\,368\,000}$$

$$\zeta^{\pm}(14) = \frac{1}{1^{14}} - \frac{1}{2^{14}} + \frac{1}{3^{14}} - \cdots = \frac{8191\cdot\pi^{14}}{74\,724\,249\,600}$$

Although ζ(1) diverges to positive and negative infinity, ζ(0) is well behaved and is known to be ζ(0) = -1/2 and the corresponding Bernoulli number is $B_0$ = -1. We can now define a rational function "$\kappa_{2k} = 2\,\zeta(2k)/\pi^{2k}$" with "$\kappa_0 = -1$" which will simplify our series as:

$$\tan x = x + \frac{x^3}{3} + \frac{2\cdot x^5}{15} + \cdots + \left(2^{2k}-1\right)\cdot\kappa_{2k}\cdot x^{2k-1} + \cdots = \frac{1}{x}\cdot\sum_{k=0}^{\infty}\left(2^{2k}-1\right)\cdot\kappa_{2k}\cdot x^{2k}$$

$$\cot x = \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \frac{2\cdot x^5}{945} - \cdots - \kappa_{2k}\cdot x^{2k-1} + \cdots = \frac{1}{x}\cdot\left(1 - \sum_{k=1}^{\infty}\kappa_{2k}\cdot x^{2k}\right) = -\sum_{k=0}^{\infty}\kappa_{2k}\cdot x^{2k-1}$$

$$\ln\sin x = \ln x - \frac{x^2}{6} - \frac{x^4}{180} - \frac{x^6}{2835} - \frac{x^8}{37800} - \cdots - \frac{\kappa_{2k}\cdot x^{2k}}{2k} - \cdots = \ln x - \sum_{k=1}^{\infty}\frac{\kappa_{2k}\cdot x^{2k}}{2k}$$

The reader can verify that we can differentiate the last equation to obtain the cot(x) equation. A closer look at the tan(x) power series reveals the identity "tan(x) = cot(x) – 2 cot(2x)" a rather impressive fact! We further conclude, that the cot(x) power series converges much faster than the tax(x) power series and is simpler in expression. By integrating the tan(x) function we can obtain the ln cos(x) function as a power series:

$$\ln\cos x = -\frac{x^2}{2} - \frac{x^4}{12} - \frac{x^6}{45} - \cdots - \frac{\left(2^{2k}-1\right)\cdot\kappa_{2k}}{2k}\cdot x^{2k} - \cdots = -\sum_{k=1}^{\infty}\frac{\left(2^{2k}-1\right)\cdot\kappa_{2k}}{2k}\cdot x^{2k}$$

The expression "$a_{2k}\,x^{2k}$" inside the sum can be defined for k=0 rendering the value "$a_0$ = -ln 2". The final result is:

$$\ln\cos x = -\ln 2 + \ln 2 - \sum_{k=1}^{\infty}\frac{\left(2^{2k}-1\right)\cdot\kappa_{2k}}{2k}\cdot x^{2k} = \ln\tfrac{1}{2} - \sum_{k=0}^{\infty}\frac{\left(2^{2k}-1\right)\cdot\kappa_{2k}}{2k}\cdot x^{2k}$$

## 8. Sinus and Gamma functions from Riemann Zeta coefficients:

We will now discover a relation linking the Gamma Function and the Sinus Function. In section 6 we explored the lnsin(x) power series which had Riemann Zeta Coefficients. By a change of variable "x = π z" and dividing by 2 it becomes:

$$\sum_{k=1}^{\infty}\frac{\zeta(2k)}{2k}\cdot z^{2k}=\sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty}\frac{1}{2k}\cdot\left(\frac{z}{\ell}\right)^{2k}=-\tfrac{1}{2}\cdot\sum_{\ell=1}^{\infty}\ln\left(1-\frac{z^2}{\ell^2}\right)=-\tfrac{1}{2}\cdot\ln\prod_{\ell=1}^{\infty}\left(1-\frac{z^2}{\ell^2}\right)=-\ln\sqrt{\frac{\sin\pi\cdot z}{\pi\cdot z}}$$

This infinite series is of even order with index 2k=2,4,6,… and can be considered as the even part of a more general series with index values k=2,3,4,5,… The odd series will accordingly have index 2k+1=3,5,7,… and it is:

$$\sum_{k=1}^{\infty}\frac{\zeta(2k+1)}{2k+1}\cdot z^{2k+1}=\sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty}\frac{1}{2k+1}\cdot\left(\frac{z}{\ell}\right)^{2k+1}=\sum_{\ell=1}^{\infty}\left[\tanh^{-1}\frac{z}{\ell}-\frac{z}{\ell}\right]=\ln\prod_{\ell=1}^{\infty}\left(\frac{\ell+z}{\ell-z}\right)^{1/2}\cdot e^{-z/\ell}$$

An interchange of summation order in the double sum revealed the Taylor series for $\tanh^{-1}(x)$. Now subtract the odd series from the even series to get a series alternating in sign:

$$\sum_{k=2}^{\infty}\frac{(-1)^k\cdot\zeta(k)}{k}\cdot z^k=\sum_{k=2}^{\infty}\sum_{\ell=1}^{\infty}\frac{(-1)^k}{k}\cdot\left(\frac{z}{\ell}\right)^k=\sum_{\ell=1}^{\infty}\left[\frac{z}{\ell}-\ln\left(1+\frac{z}{\ell}\right)\right]=-\ln\prod_{\ell=1}^{\infty}\left(1+\frac{z}{\ell}\right)\cdot e^{-z/\ell}$$

To complete this, we use Euler's Constant: $\gamma=\lim_{m\to\infty}\left[1+\tfrac{1}{2}+\tfrac{1}{3}+\cdots+\tfrac{1}{m}-\ln m\right]$ and the Gamma Function: $n!=\Gamma(n+1)$.

$$\prod_{\ell=1}^{\infty}\left(1+\frac{z}{\ell}\right)\cdot e^{-z/\ell}=\lim_{m\to\infty}\left[\prod_{k=1}^{m}e^{-z/k}\cdot\prod_{\ell=1}^{m}\left(1+\frac{z}{\ell}\right)\right]=e^{-\gamma\cdot z}\cdot\lim_{m\to\infty}\left[\frac{m^{-z}}{m!}\cdot\prod_{\ell=1}^{m}(z+\ell)\right]=\frac{e^{-\gamma\cdot z}}{\Gamma(z+1)}$$

We have thus completed the task of evaluating both the even, and the odd power series we started with, and the result is:

$$\exp\left\{\sum_{k=1}^{\infty}\frac{\zeta(2k)}{2k}\cdot z^{2k}\right\}=\prod_{\ell=1}^{\infty}\left(1-\frac{z^2}{\ell^2}\right)^{-1/2}=\sqrt{\Gamma(1+z)\cdot\Gamma(1-z)}=\sqrt{\frac{\pi\cdot z}{\sin\pi\cdot z}}$$

$$\exp\left\{\sum_{k=1}^{\infty}\frac{\zeta(2k+1)}{2k+1}\cdot z^{2k+1}\right\}=\prod_{\ell=1}^{\infty}e^{-z/\ell}\cdot\left(\frac{\ell+z}{\ell-z}\right)^{1/2}=e^{-\gamma\cdot z}\cdot\sqrt{\frac{\Gamma(1-z)}{\Gamma(1+z)}}$$

The reflective property of the Gamma Function "Γ(z) Γ(1-z) = π / sin zπ" appears here, and the odd case generates an infinite product formula to complement the even case.

## 9. The Factorial Operator and the Gamma Function:

Now we will give a rather strange expressions concerning two integers (m,n) with m>>n where (n) is fixed, but (m) will be increasing and tending towards infinity.

$$\lim_{m\to\infty}\left[m^n\cdot\frac{m!}{(m+n)!}\right]=\lim_{m\to\infty}\prod_{k=1}^{n}\frac{m}{m+k}=\lim_{m\to\infty}\prod_{k=1}^{n}\left(1+\frac{k}{m}\right)^{-1}=1$$

$$\lim_{m\to\infty}\left[m^n\cdot\frac{m!\cdot n!}{(m+n)!}\right]=\lim_{m\to\infty}\left[m^n\cdot\prod_{k=1}^{m}\frac{k}{k+n}\right]=\lim_{m\to\infty}\prod_{k=1}^{n}\frac{m\cdot k}{m+k}=\lim_{m\to\infty}\prod_{k=1}^{n}k\cdot\left(1+\frac{k}{m}\right)^{-1}=n!\equiv\Gamma(n+1)$$

The relation "$(m+n)!\geq m^n\, m!$" is an equality when n=1. To make precise the largeness of m, we do a calculus analyses and get the formula "$m>n^2/2\varepsilon$", where ε is the relative error. For example, if we want <1% for n=5, we need m>1250, a rather slow convergence! The product occurring "m" times can be used to define the general factorial function of a real or complex variable z. The result is the following definition for the Gamma Function used in section 8:

$$\Gamma(z+1)=\lim_{m\to\infty}\left[m^z\cdot\prod_{k=1}^{m}\frac{k}{k+z}\right]\quad\Rightarrow\quad\Gamma(z)=z^{-1}\cdot\lim_{m\to\infty}\left[m^z\cdot\prod_{k=1}^{m}\left(1+\frac{z}{k}\right)^{-1}\right]$$

## 10. The finite Product of sin(πk/n) and Γ(k/n):

We will now use two relationships among the Gamma Function Γ(1+z) to prove the Finite Product of Sinus for the fundamental argument (πk/n). The reflective property of the Gamma Function in section 8, will allow us to express the Sinus Function in terms of Gamma Functions. This allows the use of a known formula for Finite Products of the Gamma Function, which can be found in most standard mathematical handbooks:

$$\prod_{k=0}^{n-1}\Gamma\left(x+\frac{k}{n}\right)=(2\pi)^{(n-1)/2}\cdot n^{(1-2n\cdot x)/2}\cdot\Gamma(n\cdot x)$$

Now set "x=1" and use the recursive property of the Gamma Function "Γ(1+n) = n Γ(n)" and remember that "Γ(1)=1" to obtain:

$$\prod_{k=1}^{n-1}\Gamma\left(1+\frac{k}{n}\right)=\prod_{k=1}^{n-1}\left(\frac{k}{n}\right)\cdot\Gamma\left(\frac{k}{n}\right)=\frac{(n-1)!}{n^{n-1}}\cdot\prod_{k=1}^{n-1}\Gamma\left(\frac{k}{n}\right)=(2\pi)^{(n-1)/2}\cdot n^{(1-2n)/2}\cdot(n-1)!$$

$$\Rightarrow$$

$$\prod_{k=1}^{n-1}\Gamma\left(\frac{k}{n}\right)=(2\pi)^{(n-1)/2}\cdot n^{-1/2}\qquad\Rightarrow\qquad\prod_{k=1}^{n-1}\Gamma^2\left(\frac{k}{n}\right)=\frac{(2\pi)^{(n-1)}}{n}$$

Armed with the Finite Product of Squared Gamma Functions of the argument (k/n), we can now turn to the final proof:

$$\prod_{k=1}^{n-1}\sin\left(\frac{\pi\cdot k}{n}\right)=\prod_{k=1}^{n-1}\frac{\pi}{\Gamma\left(\frac{k}{n}\right)\cdot\Gamma\left(1-\frac{k}{n}\right)}=\pi^{n-1}\cdot\prod_{k=1}^{n-1}\frac{1}{\Gamma\left(\frac{k}{n}\right)\cdot\Gamma\left(\frac{n-k}{n}\right)}=\pi^{n-1}\cdot\prod_{k=1}^{n-1}\frac{1}{\Gamma^2\left(\frac{k}{n}\right)}=\frac{n}{2^{n-1}}$$

This was not so hard! The key to this result is recognising that "Π f(k)f(n-k) = Π $f^2$(k)" when "k=1,2…(n-1)". For example, if n=5 we get 1·4·2·3·3·2·1·4=1·1·2·2·3·3·4·4=$1^2·2^2·3^2·4^2$ which should convince the most sceptics!

## 11. Stirling's formula and lnsin(πk/n) sums:

By solving together the infinite product expansion of the sinus function and our newly obtained finite sums for lnsin(x), we can generate some powerful statements about infinite sums. Starting with the prototype argument (πk/n) we can get:

$$\sum_{k=1}^{n-1}\ln\sin\left(\frac{\pi\cdot k}{n}\right)=\sum_{k=1}^{n-1}\ln\left(\frac{\pi\cdot k}{n}\right)-\sum_{k=1}^{n-1}\sum_{\ell=1}^{\infty}\left(\frac{\zeta(2\ell)}{\ell}\right)\cdot\left(\frac{k}{n}\right)^{2\ell}$$

$$=\ln\left(\frac{(n-1)!\cdot\pi^{n-1}}{n^{n-1}}\right)-\sum_{k=1}^{n-1}\sum_{\ell=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{k}{n\cdot m}\right)^{2\ell}\cdot\ell^{-1}=\ln\left(\frac{n}{2^{n-1}}\right)$$

$$\Rightarrow$$

$$\sum_{k=1}^{n-1}\sum_{\ell=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{k}{n\cdot m}\right)^{2\ell}\cdot\ell^{-1}=\ln\left(\frac{(2\pi)^{n-1}\cdot(n-1)!}{n^n}\right)=-\tfrac{1}{2}+\left(n-\tfrac{1}{2}\right)\cdot\ln\left(\frac{2\pi}{e}\right)-\tfrac{1}{2}\cdot\ln(n)+\sigma(n)$$

Here we have used the Stirling's formula for n! = Γ(n+1) to eliminate "n!/$n^n$"! We have further taken the liberty to define a function "***σ(n) = ln(1+1/12n+1/288n²-…)***" which obviously tends fast to zero, as n grows larger.

## 12. The Factorial Triangle & Polynomials:

The core of the Gamma Function is the factor (z+k) with k=1,2,3,…n and (z) can be integer, real or complex. By performing the multiplication, a polynomial in (z) is formed. The coefficients of this polynomial can be arranged in a Pascal's-like triangle:

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | 1 | | | | | | | |
| | | | | | | 1 | | 1 | | | | | | |
| | | | | | 1 | | 3 | | 2 | | | | | |
| | | | | 1 | | 6 | | 11 | | 6 | | | | |
| | | | 1 | | 10 | | 35 | | 50 | | 24 | | | |
| | | 1 | | 15 | | 85 | | 225 | | 274 | | 120 | | |
| | 1 | | 21 | | 175 | | 735 | | 1624 | | 1764 | | 720 | |
| 1 | | 28 | | 322 | | 1960 | | 6769 | | 13132 | | 13068 | | 5040 |

Let F(n,m) represent the coefficient, where (n) is the number of factors and (m) is the coefficient's index from left to right.

***…A WORK IN PROGRESS…***

***1.may, 2002***

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