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Factoring Cubic Polynomials.

1. Introduction:
While 2nd order polynomials of the general form "$a_2\, x^2 + a_1\, x + a_0$" are easily factored by algebraic methods, 3rd order polynomials of the general form "$a_3\, x^3 + a_2\, x^2 + a_1\, x + a_0$" are rarely factorized with algebra and numerical methods dominate the art. In this article we will present a simple Theory to completely solve the most general 3rd order polynomial with real coefficients "$a_n \in \mathbb{R}$ " where "$n \in \{0,1,2,3\}$" and the independent complex variable is "$z \in \mathbb{C}$ ".

(1) $$z^3 + a_2 \cdot z^2 + a_1 \cdot z + a_0 = 0$$

Here a_0, a_1 and a_2 are real coefficients and the suppressed coefficient to the cubic term is "$a_3 = 1$". By translating z by a constant amount z_0, a new independent variable "$w = z - z_0$" is obtained and the cubic polynomial now reads:

(2) $$w^3 + \left(3 \cdot z_0 + a_2\right) \cdot w^2 + \left(3 \cdot z_0{}^2 + 2 \cdot a_2 \cdot z_0 + a_1\right) \cdot w + \left(z_0{}^3 + a_2 \cdot z_0{}^2 + a_1 \cdot z_0 + a_0\right) = 0$$

The square term can be eliminated by choosing "$z_0 = - a_2/3$", thus reducing equation (2) to a *three* term equation:

(3) $$w^3 + \left(a_1 - \tfrac{1}{3} \cdot a_2{}^2\right) \cdot w + \left(a_0 - \tfrac{1}{3} \cdot a_1 \cdot a_2 + \tfrac{2}{27} \cdot a_2{}^3\right) = 0$$

(4) $$b = \left(\tfrac{1}{3} \cdot a_1 - \tfrac{1}{9} \cdot a_2{}^2\right), \qquad c = \left(\tfrac{1}{2} \cdot a_0 - \tfrac{1}{6} \cdot a_1 \cdot a_2 + \tfrac{1}{27} \cdot a_2{}^3\right)$$

(5) $$w^3 + 3 \cdot b \cdot w + 2 \cdot c = 0$$

The mandatory real root "w_0" of the *three*-term Cubic equation (5) can be derived by algebraic means using the discriminating number "$d^2 = b^3 + c^2$" and some hyperbolic trigonometric identities as seen in the appendix:

(6) $$w_0 = -\left[\left(c + \left(c^2 + b^3\right)^{\frac{1}{2}}\right)^{\frac{1}{3}} + \left(c - \left(c^2 + b^3\right)^{\frac{1}{2}}\right)^{\frac{1}{3}}\right] = -\left[\left(c + d\right)^{\frac{1}{3}} + \left(c - d\right)^{\frac{1}{3}}\right] = -\left(p - q\right)$$

Here "$p^3 = c + d$ " and "$- q^3 = c - d$" are defined to simplify notation. The factor "$w + (p-q)$" is removed from equation (5) by division to reveal a 2nd order polynomial which gives the remaining two complex and conjugate roots:

(7) $$w_{\pm 1} = \tfrac{1}{2} \cdot \left[\left(c + d\right)^{\frac{1}{3}} + \left(c - d\right)^{\frac{1}{3}}\right] \pm i \cdot \tfrac{\sqrt{3}}{2} \cdot \left[\left(c + d\right)^{\frac{1}{3}} - \left(c - d\right)^{\frac{1}{3}}\right] = \tfrac{1}{2} \cdot \left(p - q\right) \pm i \cdot \tfrac{\sqrt{3}}{2} \cdot \left(p + q\right)$$

When "$d = 0$", we have "$p^3 = q^3 = c$" and *three real* roots appear, two of them equal: "$w_0 = -2p$" and "$w_1 = w_{-1} = p$".

2. Imaginary Discriminator:
When "$b < -c^{2/3}$", the discriminator "d" becomes imaginary. The sum and difference functions (c + d) and (c – d) now become complex conjugate pairs (c + id) and (c- id) with real magnitude $\sqrt{-b^3}$ thanks to "$b < -c^{2/3} < 0$".

(8) $$w_0 = -\left[\left(c + i \cdot d\right)^{\frac{1}{3}} + \left(c - i \cdot d\right)^{\frac{1}{3}}\right] = -\left(p + q\right) = -2\sqrt{-b} \cdot \cos\left(\tfrac{1}{3} \cdot \cos^{-1}\left(\frac{c}{\sqrt{-b^3}}\right)\right)$$

We can either use the periodicity of cosine, or factor "$w - w_0$" out from equation (5) to get the other *two real* roots as:

(9) $$w_{\pm 1} = \sqrt{-b} \cdot \left[\cos\left(\tfrac{1}{3} \cdot \cos^{-1}\left(\frac{c}{\sqrt{-b^3}}\right)\right) \pm \sqrt{3} \cdot \sin\left(\tfrac{1}{3} \cdot \cos^{-1}\left(\frac{c}{\sqrt{-b^3}}\right)\right)\right]$$

When "d" became imaginary in (6) and (7), so did "(p + q)" while "(p - q)" and "i(p + q)" were both real.

3. Charting the Cubic Discriminator: $d^2 = b^3 + c^2$

The discriminating number "d" divides the c-b plane into three distinct regions as seen in Fig 1. In the blue section "b > 0" and we have one real solution $w_0 = -A \sinh \phi$ and two complex conjugate solutions $w_1 = ½ A (\sinh \phi + i \sqrt{3} \cosh \phi)$ and $w_{-1} = ½ A (\sinh \phi - i \sqrt{3} \cosh \phi)$. In the yellow section "$-c^{2/3} < b < 0$" and we have one real solution $w_0 = -A \cosh \phi$ and two complex solutions $w_1 = ½ A (\cosh \phi + i \sqrt{3} \sinh \phi)$ and $w_{-1} = ½ A (\cosh \phi - i \sqrt{3} \sinh \phi)$. Finally, in the green section "$b < -c^{2/3}$" we have three non-equal real number solutions: $w_0 = -A \cos \theta$ and $w_1 = ½ A (\cos \theta + \sqrt{3} \sin \theta)$ and $w_1 = ½ A (\cos \theta - \sqrt{3} \sin \theta)$.

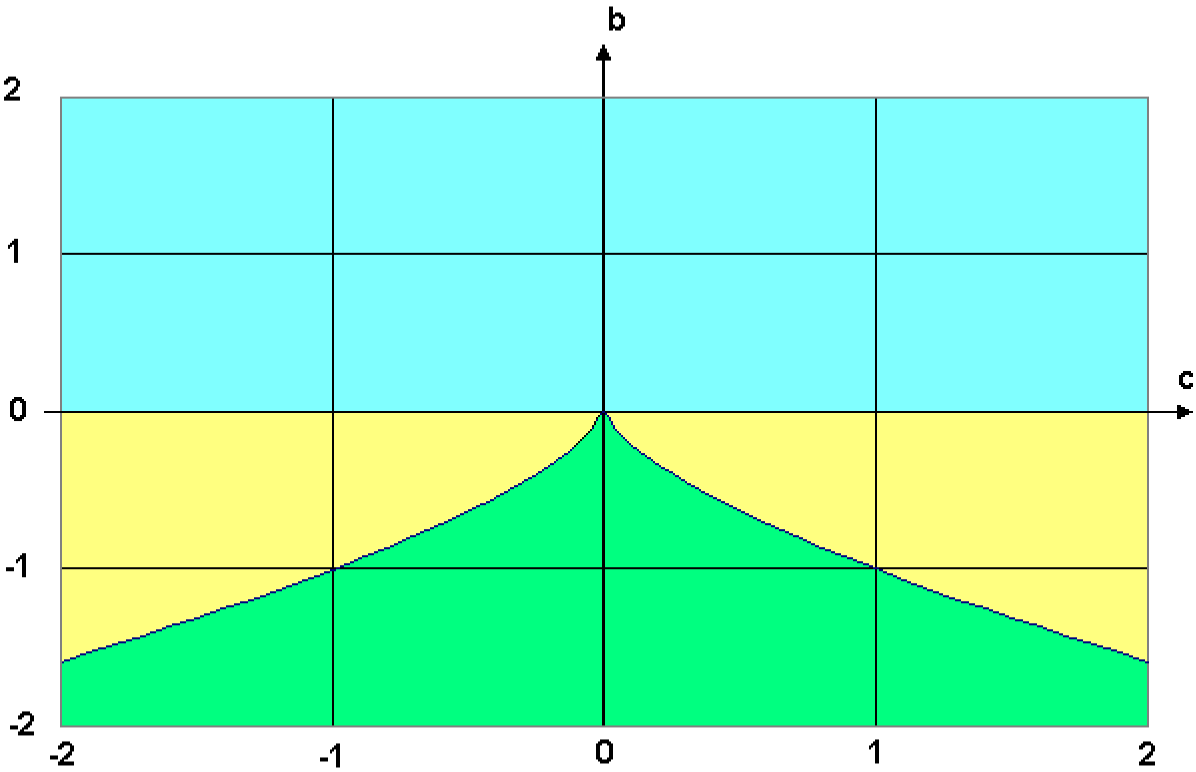

Fig 1: The c - b plane with contours $b^3 + c^2 = 0$ and $b = 0$.

4. Cubic Polynomial Theorem:

The three Roots of the equation "$w^3 + 3 \cdot b \cdot w + 2 \cdot c = 0$" belong to three regions in the 2 dimensional c-b plane. The contour sectioning the map when "b < 0" is the curves "$b^3 + c^2 = 0$" and "b = 0".

CASE I: b > 0

(10)
$$w_k = -2\sqrt{b} \cdot \sinh\left(\tfrac{1}{3} \cdot \left(2 \cdot \pi \cdot k \cdot i + \sinh^{-1}\left(\frac{c}{\sqrt{b^3}}\right)\right)\right), \quad k \in \{-1, 0, 1\}$$

NB: "b = 0" demands "$w_0 = -(2c)^{1/3}$" real but "$w_1 = (c/4)^{1/3}(1 + i \sqrt{3})$" and "$w_{-1} = (c/4)^{1/3}(1 - i \sqrt{3})$" complex.

CASE II: $-c^{2/3} < b < 0$

(11)
$$w_k = -2\sqrt{-b} \cdot \cosh\left(\tfrac{1}{3} \cdot \left(2 \cdot \pi \cdot k \cdot i + \cosh^{-1}\left(\frac{c}{\sqrt{-b^3}}\right)\right)\right), \quad k \in \{-1, 0, 1\}$$

NB: "$b = -c^{2/3}$" demands that "$w_0 = -2\, c^{1/3}$" is real and "$w_1 = w_{-1} = c^{1/3}$" is the real double root.

CASE III: $b < -c^{2/3}$

(12)
$$w_k = -2\sqrt{-b} \cdot \cos\left(\tfrac{1}{3} \cdot \left(2 \cdot \pi \cdot k + \cos^{-1}\left(\frac{c}{\sqrt{-b^3}}\right)\right)\right), \quad k \in \{-1, 0, 1\}$$

NB: "$b = -c^{2/3}$" is identical to (11), that is "$w_0 = -2 \sqrt{-b} = -2\, c^{1/3}$" and "$w_1 = w_{-1} = c^{1/3}$".

5. Details of Cubic Polynomial Theorem:
The sine hyperbolic function to the cubic root expressions "p" and "q" is dismantled here. The same can be done with cosine hyperbolic. When "b" is small, we expand the square operator and the cubic operator in terms of the parameter "b^3/c^2".

CASE I: b > 0 and c ∈ $\mathbb{R}$

$$2\sqrt{b}\cdot\sinh\left(\tfrac{1}{3}\cdot\sinh^{-1}\left(\frac{c}{\sqrt{b^3}}\right)\right)=\sqrt{b}\cdot\left[\left(\frac{c}{\sqrt{b^3}}+\sqrt{\frac{c^2}{b^3}+1}\right)^{\frac{1}{3}}-\left(\frac{c}{\sqrt{b^3}}+\sqrt{\frac{c^2}{b^3}+1}\right)^{\frac{-1}{3}}\right]$$

$$=\left[\left(c+\left(c^2+b^3\right)^{\frac{1}{2}}\right)^{\frac{1}{3}}+\left(c-\left(c^2+b^3\right)^{\frac{1}{2}}\right)^{\frac{1}{3}}\right]=\left[\left(c+d\right)^{\frac{1}{3}}+\left(c-d\right)^{\frac{1}{3}}\right]=p-q$$

CASE I: b ≥ 0 and c ≠ 0

$$p-q=\sqrt[3]{2c}\cdot\left[\cosh^{2/3}\left(\tfrac{1}{2}\cdot\sinh\tfrac{b^{3/2}}{c}\right)-\sinh^{2/3}\left(\tfrac{1}{2}\cdot\sinh\tfrac{b^{3/2}}{c}\right)\right]=\sqrt[3]{2c}\cdot\left(1-\tfrac{b^3}{12c^2}-\cdots\right)$$

$$p+q=\sqrt[3]{2c}\cdot\left[\cosh^{2/3}\left(\tfrac{1}{2}\cdot\sinh\tfrac{\sqrt{b^3}}{c}\right)+\sinh^{2/3}\left(\tfrac{1}{2}\cdot\sinh\tfrac{\sqrt{b^3}}{c}\right)\right]$$

CASE II: $-c^{2/3}$ < b < 0 and c > 0

$$2\sqrt{-b}\cdot\cosh\left(\tfrac{1}{3}\cdot\cosh^{-1}\left(\frac{c}{\sqrt{-b^3}}\right)\right)=\sqrt{-b}\cdot\left[\left(\frac{c}{\sqrt{-b^3}}+\sqrt{\frac{c^2}{-b^3}-1}\right)^{\frac{1}{3}}+\left(\frac{c}{\sqrt{-b^3}}+\sqrt{\frac{c^2}{-b^3}-1}\right)^{\frac{-1}{3}}\right]$$

$$=\left[\left(c+\left(c^2+b^3\right)^{\frac{1}{2}}\right)^{\frac{1}{3}}-\left(c-\left(c^2+b^3\right)^{\frac{1}{2}}\right)^{\frac{1}{3}}\right]=\left[\left(c+d\right)^{\frac{1}{3}}-\left(c-d\right)^{\frac{1}{3}}\right]=p+q$$

CASE I & II: b ≈ 0 and c ≠ 0

$$p-q=\sqrt[3]{c}\cdot\left[\left(1+\sqrt{1+\frac{b^3}{c^2}}\right)^{\frac{1}{3}}+\left(1-\sqrt{1+\frac{b^3}{c^2}}\right)^{\frac{1}{3}}\right]\approx\sqrt[3]{2c}-\left(\frac{1}{\sqrt[3]{2c}}\right)\cdot b+\left(\frac{\sqrt[3]{2c}}{12c^2}\right)\cdot b^3-\cdots$$

$$p+q=\sqrt[3]{c}\cdot\left[\left(1+\sqrt{1+\frac{b^3}{c^2}}\right)^{\frac{1}{3}}-\left(1-\sqrt{1+\frac{b^3}{c^2}}\right)^{\frac{1}{3}}\right]\approx\sqrt[3]{2c}+\left(\frac{1}{\sqrt[3]{2c}}\right)\cdot b+\left(\frac{\sqrt[3]{2c}}{12c^2}\right)\cdot b^3-\cdots$$

CASE II & III: b = $-c^{2/3}$ and c ≠ 0

$$p=q=\sqrt[3]{c}=\sqrt{-b}$$

Hyperbolic sine for Cubic use:

The Cubic equation in reduced form is written as "$w^3 + 3 \cdot b \cdot w + 2 \cdot c = 0$" and the trial functions for the root candidate are the hyperbolic sine and cosine, and sine and cosine themselves.

$$A^3 \cdot \sinh 3\phi - 3 \cdot A \cdot \left(A^2 - 4 \cdot b\right) \cdot \sinh \phi + 8 \cdot c = 0, \quad \Rightarrow \quad \sinh 3\phi = \frac{-c}{\sqrt{b^3}}$$

$$A^3 \cdot \cosh 3\phi + 3 \cdot A \cdot \left(A^2 + 4 \cdot b\right) \cdot \cosh \phi + 8 \cdot c = 0, \quad \Rightarrow \quad \cosh 3\phi = \frac{-c}{\sqrt{-b^3}}$$

$$-A^3 \cdot \sin 3\theta + 3 \cdot A \cdot \left(A^2 + 4 \cdot b\right) \cdot \sin \theta + 8 \cdot c = 0, \quad \Rightarrow \quad \sin 3\theta = \frac{c}{\sqrt{-b^3}}$$

$$A^3 \cdot \cos 3\theta + 3 \cdot A \cdot \left(A^2 + 4 \cdot b\right) \cdot \cos \theta + 8 \cdot c = 0, \quad \Rightarrow \quad \cos 3\theta = \frac{-c}{\sqrt{-b^3}}$$

Some identities:

$$\begin{aligned}
\sinh^3 \phi &= \tfrac{1}{4} \cdot \sinh 3\phi - \tfrac{3}{4} \cdot \sinh \phi \\
\cosh^3 \phi &= \tfrac{1}{4} \cdot \cosh 3\phi + \tfrac{3}{4} \cdot \cosh \phi \\
\sin^3 \theta &= -\tfrac{1}{4} \cdot \sin 3\theta + \tfrac{3}{4} \cdot \sin \theta \\
\cos^3 \theta &= \tfrac{1}{4} \cdot \cos 3\theta + \tfrac{3}{4} \cdot \cos \theta
\end{aligned}$$