Modified Riemann-Zeta Power Series - for Simple and Fast Calculation of the Trigonometric Functions.

Guðlaugur Kristinn Óttarsson
Pro%Nil Systems,
Flókagötu 27, 105 Reykjavik, Iceland.

ABSTRACT:

The power series for the trigonometric function *cot(x)* has Riemann-zeta-functions for coefficients - and it converges fast - thus promoting *cot(x)* to an important status in functional, numerical and computational analysis. In this paper we encounter two linear and useful expression with *cotangent* functions: "*tan(x) = cot(x) – 2 cot(2x)*" and "*csc(x) = cot(x/2) – cot(x)*" with binary scaled arguments. Four *cotangent* identities are gradually exposed or discovered throughout the paper:

$$\cot 2x = \tfrac{1}{2}\cdot\left[\cot x - \tan x\right] = \sinh\ln\cot x \qquad \text{A1}$$
$$\csc 2x = \tfrac{1}{2}\cdot\left[\cot x + \tan x\right] = \cosh\ln\cot x \qquad \text{A2}$$

$$\tan 2x = \tfrac{1}{2}\cdot\left[\tan(x-\tfrac{\pi}{4}) - \cot(x-\tfrac{\pi}{4})\right] = -\sinh\ln\cot(x-\tfrac{\pi}{4}) \qquad \text{A3}$$
$$\sec 2x = \tfrac{1}{2}\cdot\left[\tan(x-\tfrac{\pi}{4}) + \cot(x-\tfrac{\pi}{4})\right] = \cosh\ln\cot(x-\tfrac{\pi}{4}) \qquad \text{A4}$$

On the right, *sinh ln (u)* is used as a short hand for ½ $(u - u^{-1})$. The identities A1 & A2 show, that *cot x* and *csc x* are complementary to each other in the hyperbolic sense, – while A3 & A4 show, that *tan x* is complementary to *sec x*.

The fact that the double angle identities in equations A1 – A4 are linear, enables us to derive a *high-speed algorithm* to calculate all the six trigonometric functions, including *sin x* and *cos x* - for navigational, engineering, animation, signal processing, and general scientific work.

We also get a unique opportunity for a close encounter with the Riemann-zeta-function, which is inside the *cot(x)* function as a coefficient. This will leads us directly up front with the *Euler-MacLaurin Summation Formula* - which is then shown to have *cot(i/2)* as an "Eigenvalue" or "Proper value".

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Website: http://www.islandia.is/gko/papers/
Email: *gko@islandia.is*

1. Basic trigonometric functions with Riemann Zeta coefficients (or Bernoulli Numbers).

The trigonometric function *cot(x)* is rather special - being the derivative of *ln sin(x)* - and the same applies to the *tan(x)* function, which is the negative derivative of *ln cos(x)*. By expanding the *cot(x)* function and the *tan(x)* function into power series with the argument x, Bernoulli Numbers $\boldsymbol{B_k}$ or the Riemann Zeta function ζ(s) appear in coefficients in either function:

$$\cot x = \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \frac{2\cdot x^5}{945} - \cdots - \frac{2^{2k}\cdot B_k \cdot x^{2k-1}}{(2k)!} + \cdots = \frac{1}{x} - \frac{2}{x}\cdot\sum_{k=1}^{\infty}\zeta(2k)\cdot\left(\frac{x}{\pi}\right)^{2k}$$

$$\tan x = x + \frac{x^3}{3} + \frac{2\cdot x^5}{15} + \frac{17\cdot x^7}{315} + \cdots + \frac{2^{2k}\cdot\left(2^{2k}-1\right)\cdot B_k\cdot x^{2k-1}}{(2k)!} + \cdots = \frac{2}{x}\cdot\sum_{k=1}^{\infty}\left(2^{2k}-1\right)\cdot\zeta(2k)\cdot\left(\frac{x}{\pi}\right)^{2k}$$

The Riemann-Zeta function $\zeta(2k)$ is more economical and is therefore chosen for the summation. Although $\zeta(1)$ diverges to positive and negative infinity, $\zeta(0)$ is well behaved and is known to be $\zeta(0) = -1/2$ and the corresponding Bernoulli number is $\boldsymbol{B_0} = -1$. We can define an eluded rational function "$\kappa_{2k} = -2\ \zeta(2k) / \pi^{2k}$" with "$\kappa_0 = 1$" compacting the series above to:

$$\cot x = \frac{1}{x}\cdot\left(1 + \sum_{k=1}^{\infty}\kappa_{2k}\cdot x^{2k}\right) = \sum_{k=0}^{\infty}\kappa_{2k}\cdot x^{2k-1}$$

$$\tan x = -\sum_{k=0}^{\infty}\left(2^{2k}-1\right)\cdot\kappa_{2k}\cdot x^{2k-1}$$

A closer look at the power series for *tan(x)* reveals the identity "*tan(x) = cot(x) – 2 cot(2x)*" which is used to prove ***Eq* A1**. The power series for *cot(x)* converges much faster than the power series for *tax(x)* and is simpler in expression. Alternative expressions can be obtained by the methods of residues, which gives very clean and simple expansions:

$$\cot x = \sum_{-\infty}^{+\infty}\frac{1}{x+\pi\cdot\ell} = x\cdot\sum_{-\infty}^{+\infty}\frac{1}{x^2-\pi^2\cdot\ell^2} = \frac{1}{x} - \sum_{\ell=1}^{+\infty}\frac{2x}{\pi^2\cdot\ell^2-x^2} = \frac{1}{x} + \frac{\partial}{\partial x}\sum_{\ell=1}^{+\infty}\ln\left(1-\left(\frac{x}{\pi\cdot\ell}\right)^2\right)$$

$$\tan x = \sum_{-\infty}^{+\infty}\frac{1}{x+\pi\cdot(\ell-\frac{1}{2})} = x\cdot\sum_{-\infty}^{+\infty}\frac{1}{x^2-\pi^2\cdot(\ell-\frac{1}{2})^2} = -\sum_{\ell=1}^{+\infty}\frac{2x}{\pi^2\cdot(\ell-\frac{1}{2})^2-x^2} = \frac{\partial}{\partial x}\sum_{\ell=1}^{+\infty}\ln\left(1-\left(\frac{x}{\pi\cdot(\ell-\frac{1}{2})}\right)^2\right)$$

We are now in a unique position – as we can relate simplicity with complexity and gain information in the process. The first step is to integrate the right-hand equalities above - as we get both convergent power series for *ln sin(x)* and *ln cos(x)*:

$$\ln\sin x = \ln x + \sum_{\ell=1}^{+\infty}\ln\left(1-\left(\frac{x}{\pi\cdot\ell}\right)^2\right) = \ln x - \sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty}\frac{1}{k}\cdot\left(\frac{x}{\pi\cdot\ell}\right)^{2k} = \ln x - \sum_{k=1}^{\infty}\frac{\zeta(2k)}{k}\cdot\left(\frac{x}{\pi}\right)^{2k} = \ln x + \sum_{k=1}^{\infty}\kappa_{2k}\cdot\frac{x^{2k}}{2k}$$

$$\ln\cos x = \sum_{\ell=1}^{+\infty}\ln\left(1-\left(\frac{x}{\pi\cdot(\ell-\frac{1}{2})}\right)^2\right) = -\sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty}\frac{1}{k}\cdot\left(\frac{x}{\pi\cdot(\ell-\frac{1}{2})}\right)^{2k} = -\sum_{k=1}^{\infty}\frac{\left(2^{2k}-1\right)\cdot\zeta(2k)}{k}\cdot\left(\frac{x}{\pi}\right)^{2k} = \sum_{k=1}^{\infty}\cdot\kappa_{2k}^{odd}\cdot\frac{(2x)^{2k}}{2k}$$

The *Odd Kappa* is defined by the rightmost equation: "$\kappa_{2k}^{odd} = (1-2^{-2k})\ \kappa_{2k}$" and can clearly be extended to odd arguments *2k+1* as well. The *Odd Kappa* engulfs the binary factor effectively and minimally. The leftmost equations lead immediately to the infinite product formulae for both the *sin(x)* and *cos(x)* functions:

$$\sin x = x\cdot\left(1-\frac{x^2}{\pi^2}\right)\cdot\left(1-\frac{x^2}{4\pi^2}\right)\cdot\left(1-\frac{x^2}{9\pi^2}\right)\cdots = x\cdot\prod_{\ell=1}^{\infty}\left(1-\frac{x^2}{\ell^2\cdot\pi^2}\right)$$

$$\cos x = \left(1-\frac{4x^2}{\pi^2}\right)\cdot\left(1-\frac{4x^2}{9\pi^2}\right)\cdot\left(1-\frac{4x^2}{25\pi^2}\right)\cdots = \prod_{\ell=1}^{\infty}\left(1-\frac{x^2}{(\ell-\frac{1}{2})^2\cdot\pi^2}\right)$$

Using Taylor series, Residue theory, Integration and Differentiation – we have obtained both clean and powerful statements concerning the basic Trigonometric functions *sin(x)*, *cos(x)*, *cot(x)* and *tan(x)*.

2. The Kappa Function κ_m normalizes Riemann Zeta and Bernoulli numbers.

In *Section 1* we introduced *Kappa* κ as the most natural expression to convey either the presence of the Riemann-zeta functions or the Bernoulli-numbers in power series for both *cot(x)* and *tan(x)*:

$$\cot x = \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \frac{2\cdot x^5}{945} - \cdots + \kappa_{2k}\cdot x^{2k-1} + \cdots = \frac{1}{x}\cdot\left(1+\sum_{k=1}^{\infty}\kappa_{2k}\cdot x^{2k}\right) = \sum_{k=0}^{\infty}\kappa_{2k}\cdot x^{2k-1}$$

$$\tan x = x + \frac{x^3}{3} + \frac{2\cdot x^5}{15} + \frac{17\cdot x^7}{315} + \cdots - \left(2^{2k}-1\right)\cdot\kappa_{2k}\cdot x^{2k-1} - \cdots = \frac{-1}{x}\cdot\sum_{k=0}^{\infty}\left(2^{2k}-1\right)\cdot\kappa_{2k}\cdot x^{2k}$$

We restrain us from using the *Odd Kappa* "$\kappa_{2k}^{odd} = (1-2^{-2k})\,\kappa_{2k}$" in this section. The integral of the above series gives us some very clean and simple expressions of the logarithm of both *sin(x)* and *cos(x)* functions:

$$\ln\sin x = \ln x - \frac{x^2}{6} - \frac{x^4}{180} - \frac{x^6}{2835} - \frac{x^8}{37800} - \cdots + \frac{\kappa_{2k}\cdot x^{2k}}{2k} - \cdots = \ln x + \sum_{k=1}^{\infty}\frac{\kappa_{2k}\cdot x^{2k}}{2k}$$

$$\ln\cos x = -\frac{x^2}{2} - \frac{x^4}{12} - \frac{x^6}{45} - \frac{17\cdot x^8}{2520} - \cdots + \frac{\left(2^{2k}-1\right)\cdot\kappa_{2k}}{2k}\cdot x^{2k} + \cdots = \sum_{k=1}^{\infty}\frac{\left(2^{2k}-1\right)\cdot\kappa_{2k}}{2k}\cdot x^{2k}$$

The coefficient to x^{2k} inside the summation for *ln cos(x)* can be defined when *k=0* rendering the value "*ln 2*" resulting in:

$$\ln\cos x = -\ln 2 + \ln 2 + \sum_{k=1}^{\infty}\frac{\left(2^{2k}-1\right)\cdot\kappa_{2k}}{2k}\cdot x^{2k} = \ln\tfrac{1}{2} + \sum_{k=0}^{\infty}\frac{\left(2^{2k}-1\right)\cdot\kappa_{2k}}{2k}\cdot x^{2k}$$

The interval of convergence is unconditional, at least on the interval $\{-\pi < x < \pi\}$, which is in fact the largest interval to occur. For clarity let us now summarise the Riemann-zeta, Bernoulli and the present *Kappa* into a single fourfold equation:

$$\zeta(2k) = \sum_{\ell=1}^{\infty}\ell^{-2k} = \frac{(2\pi)^{2k}}{(2k)!}\cdot\frac{B_k}{2} = -\tfrac{1}{2}\cdot\pi^{2k}\cdot\kappa_{2k}$$

The second equality supports the recent attempts** to redefine the "old" Bernoulli numbers as to be even indexed. The first few Bernoulli numbers together with the *Kappa* numbers - for even arguments - are:

$$B_0=-1,\quad B_1=1/6,\quad B_2=1/30,\quad B_3=1/42,\quad B_4=1/30,\quad B_5=5/66,\quad B_6=691/2730$$
$$\kappa_0=1,\quad \kappa_2=-1/3,\quad \kappa_4=-1/45,\quad \kappa_6=-2/945,\quad \kappa_8=-2/9450,\quad \kappa_{10}=-2/93555,\quad \kappa_{12}=-1382/638512875$$

It is interesting that all the denominators in $\boldsymbol{B_n}$ above are divisible by 6 – except $\boldsymbol{B_0}$ of course – but also $\boldsymbol{B_7} = 7/6$.

3. The Odd Kappa – a further variation on the Riemann-Zeta function:

The *Odd Kappa* was introduced in Section 1 as a short hand for the *ln cos(x)* power series. We can define other variations on Kappa such as the Odd Alternating Kappa.

$$\kappa_m^{odd} = -\left(\frac{2}{\pi^m}\right)\cdot\sum_{\ell=0}^{\infty}\frac{1}{(2\ell+1)^m} = -\left(\frac{2}{\pi^m}\right)\cdot\left(1+\frac{1}{3^m}+\frac{1}{5^m}+\frac{1}{7^m}+\cdots\right) = \left(1-2^{-m}\right)\cdot\kappa_m$$

$$\kappa_m^{eve} = -\left(\frac{2}{\pi^m}\right)\cdot\sum_{\ell=1}^{\infty}\frac{1}{(2\ell)^m} = -\left(\frac{2}{2^m\cdot\pi^m}\right)\cdot\left(\frac{1}{1^m}+\frac{1}{2^m}+\frac{1}{3^m}+\cdots\right) = 2^{-m}\cdot\kappa_m$$

$$\kappa_m^{odd\pm} = \left(\frac{2}{\pi^m}\right)\cdot\sum_{\ell=0}^{\infty}\frac{(-1)^{\ell+1}}{(2\ell+1)^m} = -\left(\frac{2}{\pi^m}\right)\cdot\left(1-\frac{1}{3^m}+\frac{1}{5^m}-\frac{1}{7^m}+\cdots\right)$$

The *Odd Kappa* for even arguments has the first few values as: "$\kappa^{odd}_{(2)} = -1/4$", "$\kappa^{odd}_{(4)} = -1/48$", and "$\kappa^{odd}_{(6)} = -1/480$". The *Odd Alternating Kappa* is in fact an extension to the classical *Euler Number* $\boldsymbol{E_{n/2}}$ – but for half-integer values. The first two odd arguments, *m=1* and *m=3* gives the values "$\kappa^{odd\pm}_{(1)} = -½$" and "$\kappa^{odd\pm}_{(3)} = -1/16$" respectively.

4. Riemann Zeta and the Alternating Variant for few even arguments:

At this moment, let us pause to express the first few even Riemann Zeta values. This should be compared to the *Bernoulli Numbers* B_k and the present *Kappa* κ_{2k} up to $k = 6$ in Section 2. The *Alternating* series $\zeta^{\pm}(m)$ is also given - on the right:

$$\zeta(2)=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots=\frac{\pi^2}{6} \qquad \zeta^{\pm}(2)=\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\cdots=\frac{\pi^2}{12}$$

$$\zeta(4)=\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\cdots=\frac{\pi^4}{90} \qquad \zeta^{\pm}(4)=\frac{1}{1^4}-\frac{1}{2^4}+\frac{1}{3^4}-\cdots=\frac{7\cdot\pi^4}{720}$$

$$\zeta(6)=\frac{1}{1^6}+\frac{1}{2^6}+\frac{1}{3^6}+\cdots=\frac{\pi^6}{945} \qquad \zeta^{\pm}(6)=\frac{1}{1^6}-\frac{1}{2^6}+\frac{1}{3^6}-\cdots=\frac{31\cdot\pi^6}{30\,240}$$

$$\zeta(8)=\frac{1}{1^8}+\frac{1}{2^8}+\frac{1}{3^8}+\cdots=\frac{\pi^8}{9\,450} \qquad \zeta^{\pm}(8)=\frac{1}{1^8}-\frac{1}{2^8}+\frac{1}{3^8}-\cdots=\frac{127\cdot\pi^8}{1\,209\,600}$$

$$\zeta(10)=\frac{1}{1^{10}}+\frac{1}{2^{10}}+\frac{1}{3^8}+\cdots=\frac{\pi^{10}}{93\,555} \qquad \zeta^{\pm}(10)=\frac{1}{1^{10}}-\frac{1}{2^{10}}+\frac{1}{3^{10}}-\cdots=\frac{511\cdot\pi^{10}}{47\,900\,160}$$

$$\zeta(12)=\frac{1}{1^{12}}+\frac{1}{2^{12}}+\frac{1}{3^{12}}+\cdots=\frac{691\cdot\pi^{12}}{638\,512\,875} \qquad \zeta^{\pm}(12)=\frac{1}{1^{12}}-\frac{1}{2^{12}}+\frac{1}{3^{12}}-\cdots=\frac{1\,414\,477\cdot\pi^{12}}{1\,307\,674\,368\,000}$$

$$\zeta(14)=\frac{1}{1^{14}}+\frac{1}{2^{14}}+\frac{1}{3^{14}}+\cdots=\frac{2\cdot\pi^{14}}{18\,243\,225} \qquad \zeta^{\pm}(14)=\frac{1}{1^{14}}-\frac{1}{2^{14}}+\frac{1}{3^{14}}-\cdots=\frac{8191\cdot\pi^{14}}{74\,724\,249\,600}$$

5. Secant and Cosecant power series with *Kappa Variants* as coefficients:

The power series for the cosecant function *csc(x)* can be obtained with the identity: "*csc(x) = cot(x/2) – cot(x)*". The *Alternating Kappa* $\kappa^{\pm}$ appears in the series expansion here. The null argument evaluates to $\kappa^{\pm}_0 = -1$ which is the negative of the bare Kappa $\kappa_0 = 1$ with null argument:

$$\csc x=\sum_{k=0}^{\infty}\kappa_{2k}\cdot(x/2)^{2k-1}-\sum_{k=0}^{\infty}\kappa_{2k}\cdot x^{2k-1}=-\sum_{k=0}^{\infty}\left(1-2^{1-2k}\right)\cdot\kappa_{2k}\cdot x^{2k-1}=-\sum_{k=0}^{\infty}\kappa^{\pm}_{2k}\cdot x^{2k-1}$$

The *Alternating Kappa* is related to the *Odd* and *Even Kappa* in a simple manner: "$\kappa^{\pm} = \kappa^{odd} - \kappa^{even} = \kappa - 2\,\kappa^{eve}$" and we can easily calculate the first few Alternating Kappa values with the formula: "$\kappa^{\pm}_m = -2\,\zeta^{\pm}(m) / \pi^m$" - as follows:

$$\kappa^{\pm}_0=-1,\quad \kappa^{\pm}_2=-1/6,\ \ \kappa^{\pm}_4=-7/360,\ \ \kappa^{\pm}_6=-31/15\,120,\ \ \kappa^{\pm}_8=-127/604\,800,\ \ \kappa^{\pm}_{10}=-511/23\,950\,080$$

The *csc(x)* can be transformed into a sum of residues – and with "$z = x/2\pi$" we get the following:

$$\sum_{k=1}^{\infty}\kappa^{\pm}_{2k}\cdot x^{2k-1}=\left(\frac{2}{x}\right)\cdot\sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty}\frac{(-1)^{\ell}}{\ell^{2k}}\cdot\left(\frac{x}{\pi}\right)^{2k}=\sum_{\ell=1}^{\infty}\frac{2x\cdot(-1)^{\ell}}{\pi^2\cdot\ell^2-x^2}=z\cdot\sum_{\ell=1}^{\infty}\left[\frac{1}{\ell^2-z^2}-\frac{1}{\left(\ell-\frac{1}{2}\right)^2-z^2}\right]$$

On the right we recognize the two residue series from Section 1, which give "x^{-1} - *½ cot ½ x + ½ tan ½ x*" and is identical to "x^{-1} - *csc(x)*". Further - we can reinstate the zero-index into the right sum - to get the complete residue series for *csc(x)*.

Next we pursue the *Odd Alternating Kappa*, which is in fact an extension to the classical Euler Number $\boldsymbol{E_{n/2}}$ – but for *half-integer* values. This coefficient has the first two values $\kappa^{odd\pm}_{(1)} = -\frac{1}{2}$ and $\kappa^{odd\pm}_{(3)} = -{}^{1}/_{16}$ and "$\kappa^{odd\pm} = \kappa^{odd} - 2\kappa^{odd.even}$"

$$\sum_{k=0}^{\infty}\kappa^{odd\pm}_{2k+1}\cdot x^{2k+1}=\left(\frac{2}{x}\right)\cdot\sum_{k=0}^{\infty}\sum_{\ell=1}^{\infty}\frac{(-1)^{\ell}}{\left(\ell-\frac{1}{2}\right)^{2k+1}}\cdot z^{2k+1}=\sum_{\ell=1}^{\infty}(-1)^{\ell}\sum_{k=0}^{\infty}\left(\frac{z}{\ell-\frac{1}{2}}\right)^{2k+1}=\sum_{\ell=1}^{\infty}(-1)^{\ell}\cdot\left(\frac{z}{\ell-\frac{1}{2}}\right)\cdot\left[1-\left(\frac{z}{\ell-\frac{1}{2}}\right)^2\right]^{-1}$$

$$=z\cdot\sum_{\ell=1}^{\infty}\frac{(-1)^{\ell}\cdot\left(\ell-\frac{1}{2}\right)}{\left(\ell-\frac{1}{2}\right)^2-z^2}=\tfrac{1}{2}\cdot z\cdot\sum_{m=1}^{\infty}\left[\frac{1}{2m-\frac{1}{2}-z}+\frac{1}{2m-\frac{1}{2}+z}-\frac{1}{2m+\frac{1}{2}-z}-\frac{1}{2m+\frac{1}{2}+z}\right]$$

$$=\tfrac{1}{2}\cdot(v+w)\cdot\sum_{m=1}^{\infty}\left[\frac{v}{m^2-v^2}+\frac{w}{m^2-w^2}\right]=\tfrac{x}{4}\cdot\left[\cot(\tfrac{x}{4}+\tfrac{\pi}{4})-\cot(\tfrac{x}{4}-\tfrac{\pi}{4})\right]=-\tfrac{x}{2}\cdot\sec\tfrac{x}{2}$$

This completes the pairing of <u>four</u> Kappa Coefficients against <u>four</u> Trigonometric Functions: cot, csc, tan, sec - hence QED!

$$(\text{A1: cot}\sim\kappa)\qquad(\text{A2: csc}\sim\kappa^{\pm})\qquad(\text{A3: tan}\sim\kappa^{odd})\qquad(\text{A4: sec}\sim\kappa^{odd\pm})$$

6. Sinus and Gamma functions from series with Riemann Zeta coefficients:

We will now discover a relation linking the Gamma Function and the Sinus Function. In section 6 we explored the *ln sin(x)* power series with Riemann Zeta connection. By a change of variable "$x = \pi z$" and dividing by 2 it becomes:

$$\sum_{k=1}^{\infty}\frac{\zeta(2k)}{2k}\cdot z^{2k} = \sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty}\frac{1}{2k}\cdot\left(\frac{z}{\ell}\right)^{2k} = -\tfrac{1}{2}\cdot\sum_{\ell=1}^{\infty}\ln\left(1-\frac{z^2}{\ell^2}\right) = -\tfrac{1}{2}\cdot\ln\prod_{\ell=1}^{\infty}\left(1-\frac{z^2}{\ell^2}\right) = -\ln\sqrt{\frac{\sin\pi\cdot z}{\pi\cdot z}}$$

This infinite series is of even order with index *2k=2,4,6,...* and can be considered as the even part of a more general series with index values *k=2,3,4,5,…* The odd series will accordingly have index *2k+1=3,5,7,…* and it is:

$$\sum_{k=1}^{\infty}\frac{\zeta(2k+1)}{2k+1}\cdot z^{2k+1} = \sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty}\frac{1}{2k+1}\cdot\left(\frac{z}{\ell}\right)^{2k+1} = \sum_{\ell=1}^{\infty}\left[\tanh^{-1}\frac{z}{\ell}-\frac{z}{\ell}\right] = \ln\prod_{\ell=1}^{\infty}\left(\frac{\ell+z}{\ell-z}\right)^{1/2}\cdot e^{-z/\ell}$$

An interchange of summation order in the double sum revealed the Taylor series for *arctanh(x)*. Now subtract the odd series from the even series to get a series alternating in sign:

$$\sum_{k=2}^{\infty}\frac{(-1)^k\cdot\zeta(k)}{k}\cdot z^k = \sum_{k=2}^{\infty}\sum_{\ell=1}^{\infty}\frac{(-1)^k}{k}\cdot\left(\frac{z}{\ell}\right)^k = \sum_{\ell=1}^{\infty}\left[\frac{z}{\ell}-\ln\left(1+\frac{z}{\ell}\right)\right] = -\ln\prod_{\ell=1}^{\infty}\left(1+\frac{z}{\ell}\right)\cdot e^{-z/\ell}$$

To complete this, we use Euler's Constant: $\gamma = \lim_{m\to\infty}\left[1+\tfrac{1}{2}+\tfrac{1}{3}+\cdots+\tfrac{1}{m}-\ln m\right]$ and the Gamma Function: $n! = \Gamma(n+1)$.

$$\prod_{\ell=1}^{\infty}\left(1+\frac{z}{\ell}\right)\cdot e^{-z/\ell} = \lim_{m\to\infty}\left[\prod_{k=1}^{m}e^{-z/k}\cdot\prod_{\ell=1}^{m}\left(1+\frac{z}{\ell}\right)\right] = e^{-\gamma\cdot z}\cdot\lim_{m\to\infty}\left[\frac{m^{-z}}{m!}\cdot\prod_{\ell=1}^{m}(z+\ell)\right] = \frac{e^{-\gamma\cdot z}}{\Gamma(z+1)}$$

We have thus completed the task of evaluating both the even, and the odd power series we started with, and the result is:

$$\exp\left\{\sum_{k=1}^{\infty}\frac{\zeta(2k)}{2k}\cdot z^{2k}\right\} = \prod_{\ell=1}^{\infty}\left(1-\frac{z^2}{\ell^2}\right)^{-1/2} = \sqrt{\Gamma(1+z)\cdot\Gamma(1-z)} = \sqrt{\frac{\pi\cdot z}{\sin\pi\cdot z}}$$

$$\exp\left\{\sum_{k=1}^{\infty}\frac{\zeta(2k+1)}{2k+1}\cdot z^{2k+1}\right\} = \prod_{\ell=1}^{\infty}e^{-z/\ell}\cdot\left(\frac{\ell+z}{\ell-z}\right)^{1/2} = e^{-\gamma\cdot z}\cdot\sqrt{\frac{\Gamma(1-z)}{\Gamma(1+z)}}$$

The reflective property of the Gamma Function "$\Gamma(z)\,\Gamma(1-z) = \pi / \sin z\pi$" appears here, and the odd case generates an infinite product formula to complement the even case. We can now write: " $\csc x = \pi^{-1}\,\Gamma(x/\pi)\,\Gamma(1-x/\pi)$ " and compare with ***Eq. A2***.

7. The Euler - MacLaurin Summation formula with *Kappa* Coefficients.

In sections 1 and 2 we have elaborated on the trigonometric functions *cot(x)* and *tan(x)* and found very economical power series expansions for them – as well as an infinite product expansion for both *sin(x)* and *cos(x)*, In the present section we will discover that the *cot(x)* power series is in fact a prototype for the most general sum $\Sigma f(k)$ and in the process relate the Euler-MacLaurin summation formula to our newly obtained *cot(x)* power series. In fact the Euler- MacLaurin summation formula is shown to be a weighted *cotangent* power series for the argument *(i/2)*.

$$\sum_{k=0}^{n}f(k) = \tfrac{1}{2}\left[f(0)+f(n)\right]+\int_0^n\left[\sum_{m=0}^{\infty}\kappa_{2m}\cdot\left(\frac{i}{2}\right)^{2m}\cdot f^{(2m)}(s)\right]\cdot ds$$

Here $f^{(2m)}$ symbolises the even derivative of the function *f(s)* and the zero derivative is the function itself $f^{(0)} = f$. To take an explicit example let $f(s) = e^s$ which gives the simple result that $f^{(m)} = f$ - which effectively eliminates all even derivatives from the summation. This will give:

$$\sum_{k=0}^{n}e^k = \tfrac{1}{2}\cdot(e^n+1)+\tfrac{1}{2}\cdot(e^n-1)\cdot\sum_{m=0}^{\infty}\kappa_{2m}\cdot\left(\frac{i}{2}\right)^{2m} = \tfrac{1}{2}\cdot(e^n+1)+\tfrac{1}{2}\cdot(e^n-1)\cdot\coth\tfrac{1}{2} = \cdots = \frac{e^{n+1}-1}{e-1}$$

The last equality is identical to that calculated directly from the Geometric Series $\Sigma x^k = (x^{n+1}-1)/(x-1)$.

8. Inverted Euler - MacLaurin Summation formula normalized with *Kappa*:

By differentiating the Euler-MacLaurin summation formula and use the *Even Kappa* as a coefficient we get very efficient and clean expression for the reversed problem – that is - how to calculate an infinite series with even terms:

$$\sum_{m=0}^{\infty} f^{(2m)}(n)\cdot\kappa_{2m}\cdot\left(\frac{i}{2}\right)^{2m} = \tfrac{1}{2}\cdot\frac{\partial}{\partial n}f(n)+\frac{\partial}{\partial n}\sum_{k=1}^{n-1}f(k)$$

We shall now test this formula with the exponential function $f(n) = e^n$ as before with the Geometric Series.

$$e^n\cdot\sum_{m=0}^{\infty}\kappa_{2m}\cdot\left(\frac{i}{2}\right)^{2m} = \tfrac{1}{2}\cdot e^n+\frac{\partial}{\partial n}\sum_{k=1}^{n-1}e^k = \tfrac{1}{2}\cdot e^n+\frac{\partial}{\partial n}\left(\frac{e^n-e}{e-1}\right)=\tfrac{1}{2}\cdot e^n\cdot\left(\frac{e+1}{e-1}\right)$$

Dividing out e^n we have the familiar *cot(x)* series from section 1 and 2:

$$\sum_{m=0}^{\infty}\kappa_{2m}\cdot\left(\frac{i}{2}\right)^{2m} = \tfrac{1}{2}\cdot\left(\frac{e+1}{e-1}\right)=\tfrac{1}{2}\cdot\coth\tfrac{1}{2}$$

Now curiosity arises about the odd sum – and using results from section 5, we have the following odd series as a reference:

$$\sum_{k=1}^{\infty}\frac{\kappa_{2k+1}}{2k+1}\cdot x^{2k+1} = \frac{2\cdot\gamma\cdot x}{\pi}+\ln\Gamma\left(1+\frac{x}{\pi}\right)-\ln\Gamma\left(1-\frac{x}{\pi}\right)=$$

Here (γ) is Euler's Constant "$\gamma = 0.577215664901532\ldots$" and by performing a term by term differentiation, the following is obtained:

$$\sum_{k=1}^{\infty}\kappa_{2k+1}\cdot x^{2k+1} = \left(\frac{x}{\pi}\right)\cdot\left[2\gamma+\frac{\Gamma'\left(1+\frac{x}{\pi}\right)}{\Gamma\left(1+\frac{x}{\pi}\right)}+\frac{\Gamma'\left(1-\frac{x}{\pi}\right)}{\Gamma\left(1-\frac{x}{\pi}\right)}\right]$$

We now apply results from the standard literature which expresses Γ'/Γ as an infinite series of harmonic differences – and with the extra benefit to cancel Euler's Constant above to obtain:

$$\sum_{k=1}^{\infty}\kappa_{2k+1}\cdot x^{2k+1} = x\cdot\sum_{\ell=1}^{\infty}\left(\frac{2}{\pi\cdot\ell}-\frac{1}{\pi\cdot\ell+x}-\frac{1}{\pi\cdot\ell-x}\right)=\left(\frac{2x}{\pi}\right)\cdot\sum_{\ell=1}^{\infty}\left(\frac{1}{\ell}-\frac{\ell}{\ell^2-(x/\pi)^2}\right)$$

If we compare this with Section 1 - which gave *cot(x)* and *tan(x)* as an infinite harmonic series – we have accomplishment the analogous reduction for the odd case. Summing odd and even terms, starting from *k=2*, gives the following three identities:

$$\left(\frac{\pi}{2}\right)\cdot\sum_{k=2}^{\infty}\kappa_k\cdot x^{k-1} \;=\; \gamma+\frac{\Gamma'\left(1-\frac{x}{\pi}\right)}{\Gamma\left(1-\frac{x}{\pi}\right)} \;=\; \sum_{\ell=1}^{\infty}\left(\frac{1}{\ell}-\frac{1}{\ell-(x/\pi)}\right)$$

Based on the prototype functions above with *Kappa* coefficients – the extended inverted Euler-MacLaurin summation is:

$$\sum_{\substack{m=0\\ m\neq 1}}^{\infty} f^{(m)}(n)\cdot\kappa_m\cdot\left(\frac{i}{2}\right)^{m} = \frac{\partial}{\partial n}\left(\sum_{k=1}^{n}f(k)\right) \;-\; \tfrac{1}{2}\cdot\frac{\partial}{\partial n}f(n) \;+\; i\cdot\{\,imaginary\ \ part\,\}$$

The real part corresponds to the Inverted Euler-MacLaurin summation. Beware that *m=1* is excluded from the sum. But on the other hand – if we define "$\kappa_1 = -i$" then it is possible to move $\delta f/\delta n$ into the left sum – a tempting thought!

9. Investigating the Imaginary part of Euler - MacLaurin:

Now insert simple test functions like k, k^2, k^3 and k^3 into the general sum – and with some work we get:

$f(k)$	Re	Im
k	$n = \left(n + \frac{1}{2}\right) - \frac{1}{2}$	
k^2	$n^2 + \frac{1}{6} = \left(n^2 + n + \frac{1}{6}\right) - n$	
k^3	$n^3 + \frac{1}{2} \cdot n = \left(n^3 + \frac{3}{2} \cdot n^2 + \frac{1}{2} \cdot n\right) - \frac{3}{2} \cdot n^2$	$-\frac{3}{4} \cdot \kappa_3 = \cdots$
k^4	$n^4 + n^2 - \frac{1}{30} = \left(n^4 + \frac{2}{1} \cdot n^3 + n^2 - \frac{1}{30}\right) - \frac{2}{1} \cdot n^3$	$-3n \cdot \kappa_3 = \cdots$

The imaginary part is a slow starter and we must go much higher to gain sight – but we could hope for a recursive relation among the *Odd Kappa* – or equivalently Riemann-Zeta for an odd argument. Before we go further into the imaginary part we need more equipment – the subject of the next section.

10. Binary Structures and Logic Relationships in the Kappa Coefficients:

Now let us summarise the relationship among the *Kappa Coefficients* encountered – now including the Euler numbers:

$$\kappa^{odd} = \kappa - \kappa^{even} = (1 - 2^{-m})\, \kappa_m$$

$$\kappa^{\pm} = \kappa^{odd} - \kappa^{even} = \kappa - 2\, \kappa^{even} = (1 - 2^{1-m})\, \kappa_m$$

$$\kappa^{odd\pm} = \kappa^{odd.odd} - \kappa^{odd.even} = \kappa^{odd} - 2\kappa^{odd.even}$$

These functions generate the four basic “polar” trigonometric functions cot, csc, tan & sec respectively, as a power series:

$$x \cdot \cot x = \sum_{k=0}^{\infty} \kappa_{2k} \cdot x^{2k} \qquad x \cdot \csc x = -\sum_{k=0}^{\infty} \kappa_{2k}^{\pm} \cdot x^{2k}$$

$$x \cdot \tan x = -\sum_{k=0}^{\infty} \kappa_{2k}^{odd} \cdot (2x)^{2k} \qquad x \cdot \sec x = -\sum_{k=0}^{\infty} \kappa_{2k+1}^{odd\pm} \cdot (2x)^{2k+1}$$

We see that a family of functions are emerging containing the Trigonometric functions and the Gamma function.